Model Categories by Example

Lecture 2

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Recap

A model category is a category $\mathcal{C}$ with 3 classes of morphisms

- weak Equivs
- Fibrations
- Cofibrations

Satisfied MC1 - MC5

1) Small limits + colims
2) 2-out-of-3 for weak Equiv
3) retracts
4) lifting prop
5) factorization

Recap

Introduced a homotopy relation $X \sim Y$.

$\mathcal{C}_{cf} \subseteq \mathcal{C}$

bifibrant objects

$X \sim Y \iff X \xrightarrow{\sim} Y$

$Ho(\mathcal{C}) = \mathcal{C}_{cf}/\sim \;\simeq \mathcal{C}[W^{-1}]$

Ex

$Top_{Quillen}$

Cat_{Nat}

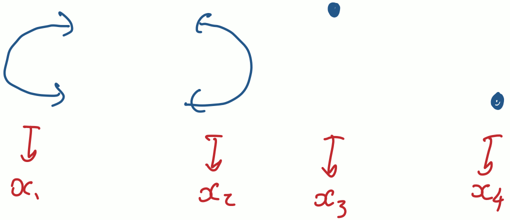
A non-invertible weak equivalence $x \xrightarrow{\sim} y$

The *pseudocircle* is the topological space $\mathbb{S}$ whose underlying set is the quadruple $\{x_1, x_2, x_3, x_4\}$ whose open subsets are

$$\{\{x_1,x_2,x_3,x_4\},\{x_1,x_2,x_3\},\{x_1,x_2,x_4\},\{x_1,x_2\},\{x_1\},\{x_2\},\varnothing\}$$

$X \simeq Y$

$f : S^1 \longrightarrow \mathbb{S}$



f is continuous + weak hmpty equiv. $f : S^1 \xrightarrow{\sim} \mathbb{S}$ in $\mathrm{Top}_{\mathrm{Quillen}}$

But the only cts function $g : \mathbb{S} \to S^1$ is constant!

(Co)limits in the homotopy category

Proposition: Let $\mathcal{C}$ be a model category, then $\mathrm{Ho}(\mathcal{C})$ has all small products and coproducts.

However, in general $\mathrm{Ho}(\mathcal{C})$ does not have all small limits and colimits.

(Co)limits in the homotopy category

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $f\colon S^1 \to S^1$ such that $f(z) = z^2$. Consider the span

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d] & S^1 \arrow[d, dotted] \\ * \arrow[r, dashed] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \Rightarrow \forall X,\ [P, X] \cong \underbrace{\mathrm{Hom}(\mathbb{Z}/2, \pi_1(X))}_{\text{picks out 2-torsion}}$

$\exists$ a fibration of spaces $\quad S^1 = B\mathbb{Z} \xrightarrow{f} S^1 \to \mathbb{R}P^\infty = B(\mathbb{Z}/2) \to \mathbb{C}P^\infty = BS^1 \xrightarrow{Bf} \mathbb{C}P^\infty$

$\Rightarrow$ exact sequence $\quad [P, S^1] \to [P, \mathbb{R}P^\infty] \to [P, \mathbb{C}P^\infty]$

$\Rightarrow$ exact sequence $\quad 0 \to \mathbb{Z}/2 \to 0 \quad$ ↯

Comparing homotopy theories

Idea Have 2 model categories $\mathcal{C}$, $\mathcal{D}$.

Functors on the underlying categories $F: \mathcal{C} \to \mathcal{D}$

Question when do we get a "derived functor" on the htpy categories?

$$\mathbb{F}: \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$$

Quillen functors

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. An adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a *Quillen adjunction* if:

- $\Leftrightarrow$ F preserves cofibrations and acyclic cofibrations;
- $\Leftrightarrow$ U preserves fibrations and acyclic fibrations;
- $\Leftrightarrow$ F preserves cofibrations and U preserves fibrations;
- $\Leftrightarrow$ F preserves acyclic cofibrations and U preserves acyclic fibrations.

Ken Brown's Lemma: Given a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ then

- F preserves weak equivalences between cofibrant objects
- U preserves " " fibrant objects.

Quillen functors

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction, define

- The *left derived functor* of F to be the composite

$$\mathbb{F}\colon \operatorname{Ho}(\mathcal{C}) \xrightarrow{\operatorname{Ho}(Q)} \operatorname{Ho}(\mathcal{C}) \xrightarrow{\operatorname{Ho}(F)} \operatorname{Ho}(\mathcal{D}).$$

- The *right derived functor* of U to be the composite

$$\mathbb{U}\colon \operatorname{Ho}(\mathcal{D}) \xrightarrow{\operatorname{Ho}(R)} \operatorname{Ho}(\mathcal{D}) \xrightarrow{\operatorname{Ho}(U)} \operatorname{Ho}(\mathcal{C}).$$

Proposition: $\mathbb{F} : \operatorname{Ho}(\mathcal{C}) \rightleftarrows \operatorname{Ho}(\mathcal{D}) : \mathbb{U}$ is an adjoint pair.

Unbounded chain complexes

Let R be a ring an $\mathbf{Ch}(R)$ the category of unbounded chain complexes of R-modules.

This admits a model structure

- W = quasi isos
- Fib = degreewise epimorphisms
- cof if it is degree wise split inj with projective cokernel $LLP(\text{acy fib})$

$$Ho(Ch(R)_{proj}) \simeq D(R)$$

we call this the projective model structure $Ch(R)_{proj}$

Unbounded chain complexes

Let X be a cofibrant object in $Ch(R)_{proj}$

[Assume R is commutative]

$Hom_R(X, -) : Ch(R)_{proj} \to Ch(R)_{proj}$ is a right Quillen

functor. Left adjoint $X \otimes_R -$

$\mathbb{R}Hom_R(X, -) = Ext(X, -)$

$\mathbb{L}(X \otimes_R -) = Tor(X, -)$

Quillen equivalences

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories equipped with a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$, then $\mathcal{C}$ and $\mathcal{D}$ are *Quillen equivalent* if the derived adjunction $\mathbb{F} : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{U}$ is an equivalence of categories.

Proposition: An adjoint pair is a Quillen equiv iff for all cofibrant $X \in \mathcal{C}$, and fibrant $Y \in \mathcal{D}$, a morphism $f : FX \to Y$ is a weak eq in $\mathcal{D}$ iff $\varphi(f) : X \to UY$ is a weak equiv in $\mathcal{C}$.

* Fact Quillen equiv satisfy 2-out-of-3

Stable module categories

- A ring R is *Frobenius* is the projective and injective R-modules coincide.

- Maps $f, g\colon M \to N$ in R-modules are *stably equivalent* if $f - g$ factors through a projective module.

<u>Prop</u> R a Frobenius ring. There is a model structure on R-mod

- W = stable equivs
- Fib = surjection
- Cof = injections

$$Ho(R\text{-mod}_{st}) =: \mathrm{StMod}(R)$$

All objects are bifibrant.

A non-Quillen equivalence (Exotic models)

Schlichting

Proposition: Let p be an odd prime, $R = \mathbb{Z}/p^2$ and $S = (\mathbb{Z}/p)[\varepsilon]/(\varepsilon^2)$.

$$\mathrm{Ho}(R\text{-mod}_{st}) \simeq \mathrm{Ho}(S\text{-mod}_{st})$$

<u>But</u> They are <u>not</u> Quillen Equivalent.

Model structures on **Set**

Fact: There are exactly nine model structures on the category **Set** of sets and functions between them.

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- bij = bijections
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- surj = surjections
- all = all morphisms
- $\mathrm{inj}_{\varnothing}$ = injections w/ empty domain
- $\mathrm{inj}_{\neq\varnothing}$ = injections w/ non-empty domain
- $\mathrm{all}_{\neq\varnothing}$ = morphisms w/ non-empty domain

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(W, Fib, Cof)

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$(7) \xrightarrow[\sim]{id} (8)$ left Quillen

$\mathrm{Ho}(\mathbf{Set}^a_{(-1)}) \simeq *$ $*$

(1) $\mathbf{Set}^a_{(-2)} = (\mathrm{all}, \mathrm{all}, \mathrm{bij})$

(2) $\mathbf{Set}^b_{(-2)} = (\mathrm{all}, \mathrm{inj}, \mathrm{surj})$

(3) $\mathbf{Set}^c_{(-2)} = (\mathrm{all}, \mathrm{surj} \cup \mathrm{inj}_\varnothing, \mathrm{inj}_{\neq\varnothing} \cup \{\mathrm{id}_\varnothing\})$

(4) $\mathbf{Set}^d_{(-2)} = (\mathrm{all}, \mathrm{surj} \cup \mathrm{bij}_\varnothing, \mathrm{all}_{\neq\varnothing} \cup \{\mathrm{id}_\varnothing\})$

(5) $\mathbf{Set}^e_{(-2)} = (\mathrm{all}, \mathrm{surj}, \mathrm{inj})$

(6) $\mathbf{Set}^f_{(-2)} = (\mathrm{all}, \mathrm{bij}, \mathrm{all})$

(7) $\mathbf{Set}^a_{(-1)} = (\mathrm{all}_{\neq\varnothing} \cup \{\mathrm{id}_\varnothing\}, \mathrm{surj} \cup \mathrm{inj}_\varnothing, \mathrm{inj})$

(8) $\mathbf{Set}^b_{(-1)} = (\mathrm{all}_{\neq\varnothing} \cup \{\mathrm{id}_\varnothing\}, \mathrm{bij} \cup \mathrm{inj}_\varnothing, \mathrm{all})$

(9) $\mathbf{Set}^a_{(0)} = (\mathrm{bij}, \mathrm{all}, \mathrm{all})$

Model structures on **Set**

$$\begin{tikzcd}
 & b \arrow[ddl] \arrow[loop above] & \\
a \arrow[ur] \arrow[rr] \arrow[drr] \arrow[d] \arrow[ddr] & & c \arrow[dll] \arrow[d] \arrow[ddl] \\
f & & d \arrow[ll] \\
 & e \arrow[ul] \arrow[loop below] &
\end{tikzcd}$$

$\mathbf{Set}_{(-2)}$

$\longrightarrow$ = id functor being left Quillen.

There is no direct Quillen equivalence!

$b \xrightarrow{\sim} f \xleftarrow{\sim} e$

Sometimes we need a zig-zag

Simplicial sets

Δ category with objects $[n] = \{0 < 1 < \cdots < n\}$.
morphisms are order preserving maps "Simplex category"

Def^n A simplicial set is a functor $X_\bullet : \Delta^{op} \to \mathrm{Set}$

sSet : Category of simp. sets + natural trans^f. $\mathrm{Set}^{\Delta^{op}}$

$\mathcal{C}$ a category $s\mathcal{C}$ is $\mathcal{C}^{\Delta^{op}}$

$X_\bullet \in \mathrm{sSet}$ this the data of
$X_n := X([n])$

face maps $d_i : X_n \to X_{n-1}$
degeneracy maps $s_i : X_n \to X_{n+1}$
$0 \leq i \leq n$

Ex $\mathcal{C} \in \mathrm{Cat}$ define its nerve $N(\mathcal{C}) \in \mathrm{sSet}$.

s_i inserts identity at pos i

d_i composes i^{th} + $(i+1)^{st}$ arrows

$(N\mathcal{C})_0 = \mathrm{ob}(\mathcal{C})$

$(N\mathcal{C})_1 = \mathrm{mor}(\mathcal{C})$

$\vdots$

$(N\mathcal{C})_k = \{$strings of k composable arrows in $\mathcal{C}\}$

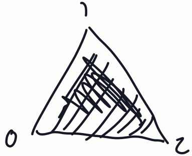
$\tau_1 : \mathrm{sSet} \rightleftarrows \mathrm{Cat} : N$.

Simplicial sets

By Yoneda, have representable objects $\Delta[n]$

$$\Delta[n]_m = \mathrm{Hom}_\Delta([m],[n]).$$

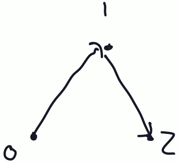
$\Delta[2]$:



$0 \le k \le n$ $\quad \Lambda^k[n] \subset \Delta[n]$

(k,n) - "horn" $\rightsquigarrow$ delete face opposite vertex k.

$\Lambda^1[2]$ =



X is a Kan complex if

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, dotted]
\end{tikzcd}$$

$\forall n \geq 0$

$0 \le k \le n$

Simplicial sets

<u>Prop</u> sSet has a model structure where:

- Fibrat objects are the Kan complexes
- Cofibration are the monomorphisms.

$sSet_{Kan}$

Simplicial sets

Proposition: There is a model structure on **sSet** where:

- The fibrant objects are
- The cofibrations are

Simplicial sets as spaces

Let $X \in \mathrm{Top}$. The singular complex of X to be the simp. set $S(X)$.

$$S(X)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta_n, X)$$

topological n-simplex

<u>Thm</u> $S(-) : \mathrm{Top}_{\mathrm{Quillen}} \rightleftarrows \mathrm{sSet}_{\mathrm{Kan}} : |-|$ is a Quillen Equivalence.

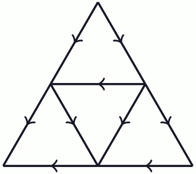
$$\Rightarrow \mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq \mathrm{Ho}(\mathrm{sSet}_{\mathrm{Kan}})$$

$f: X \to Y$ in $\mathrm{sSet}_{\mathrm{Kan}}$ is a weak equiv iff $|f| : |X| \to |Y|$ are weak eq in $\mathrm{Top}_{\mathrm{Quillen}}$

Kan's Ex^∞ functor

The *barycentric subdivision* is a left adjoint $\mathrm{sd}\colon \mathbf{sSet} \to \mathbf{sSet}$.

$\mathrm{lv}\colon \mathrm{sd}\,\Delta[n] \to \Delta[n]$
$(i_0, \dots, i_m) \mapsto (i_m)$

$\mathrm{sd}(\Delta[2]) =$ 

$\mathrm{Ex}(X)_n = \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{sd}\,\Delta[n], X)$ the right adjoint to sd.

$$X \xrightarrow{j_X} \mathrm{Ex}(X) \longrightarrow \mathrm{Ex}^2(X) \to \dots$$

colimit of this system is $\mathrm{Ex}^\infty(X)$

<u>Prop</u> $\mathrm{Ex}^\infty(X)$ is a Kan complex, $X \to \mathrm{Ex}^\infty(X)$ is an acyclic cofibration.